A Note on the Expression Complexity of Bounded Variable Fragments of the Modal μ-Calculus

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Abstract

We show by a simple reduction to reachability games, inspired by the translation of the modal μ-calculus into MSO, that the model checking problem for every bounded-variable fragment of the modal μ-calculus on a fixed transition system is in deterministic quadratic time.

1 The Modal μ-Calculus

Let $\mathcal{P}$ be a set of atomic propositions, $\mathcal{A}$ a finite set of action names, and $\mathcal{V}$ be a set of second-order variables. Formulas of the modal μ-calculus $\mathcal{L}_\mu$ are defined by the following grammar.

$$\varphi \ ::= \ q \mid X \mid \varphi \vee \varphi \mid \neg\varphi \mid \langle a\rangle\varphi \mid \nu X.\varphi$$

where $q \in \mathcal{P}$, $a \in \mathcal{A}$, and $X \in \mathcal{V}$. We furthermore assume that every variable X occurs under an even number of negation symbols in its defining fixpoint formula $\sigma X.\varphi$.

The Fischer-Ladner closure of φ is the least set $FL(\varphi)$ that contains φ and satisfies the following.

- If $\psi_1 \vee \psi_2 \in FL(\varphi)$ then $\{\psi_1, \psi_2\} \subseteq FL(\varphi)$.
- If $\neg(\psi_1 \vee \psi_2) \in FL(\varphi)$ then $\{\neg\psi_1, \neg\psi_2\} \subseteq FL(\varphi)$.
- If $\langle a\rangle\psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$.
- If $\neg\langle a\rangle\psi \in FL(\varphi)$ then $\neg\psi \in FL(\varphi)$.
- If $\neg\neg\psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$.
- If $\nu X.\psi \in FL(\varphi)$ then $\psi \in FL(\varphi)$.
- If $\neg\nu X.\psi \in FL(\varphi)$ then $\neg\psi \in FL(\varphi)$.

The size of the Fischer-Ladner closure of a formula φ is at most twice its length. Hence, we simply define $|\varphi| := |FL(\varphi)|$.

We also consider $var(\varphi) := |\mathcal{V} \cap FL(\varphi)|$, the number of variables in a formula, as a structural measure for its complexity. Let $\mathcal{L}_\mu^k := \{\varphi \mid var(\varphi) \leq k\}$ for each $k \in \mathbb{N}$ be the fragment of the modal μ-calculus that is obtained by restricting the number of different variables in each formula to at most k.

The modal μ-calculus is interpreted over labeled transition systems $\mathcal{T} = (\mathcal{S}, \{\xrightarrow{a} \mid a \in \mathcal{A}\}, L)$ where $\mathcal{S}$ is a set of state, $\xrightarrow{a} \subseteq \mathcal{S} \times \mathcal{S}$ is a binary modal accessibility relation on states for each $a \in \mathcal{A}$, and $L : \mathcal{P} \rightarrow 2^{\mathcal{S}}$ assigns to each atomic proposition the states that are satisfied by it.

Let φ_0 be fixed. In order to interpret subformulas $\psi \in FL(\varphi_0)$ we use environments $\rho : \mathcal{V} \cap FL(\varphi_0) \rightarrow 2^{\mathcal{S}}$.

$$
\begin{array}{lcl}
[\![q]\!]_\rho^\mathcal{T} & := & L(q) \\
[\![X]\!]_\rho^\mathcal{T} & := & \rho(X) \\
[\![\psi_1 \vee \psi_2]\!]_\rho^\mathcal{T} & := & [\![\psi_1]\!]_\rho^\mathcal{T} \cup [\![\psi_2]\!]_\rho^\mathcal{T} \\
[\![\neg\psi]\!]_\rho^\mathcal{T} & := & \mathcal{S} \setminus [\![\psi]\!]_\rho^\mathcal{T} \\
[\![\langle a\rangle\psi]\!]_\rho^\mathcal{T} & := & \{\, s \in \mathcal{S} \mid \exists t \in \mathcal{S} : s \xrightarrow{a} t \text{ and } t \in [\![\psi]\!]_\rho^\mathcal{T} \,\} \\
[\![\nu X.\psi]\!]_\rho^\mathcal{T} & := & \bigcup \{\, T \subseteq \mathcal{S} \mid T \subseteq [\![\psi]\!]_{\rho[X \mapsto T]}^\mathcal{T} \,\}
\end{array}
$$

where $\rho[X \mapsto T]$ denotes the environment that maps X to T and agrees with ρ an all other arguments.

Lemma 1 *For all φ, all transition systems $\mathcal{T}$ with state set $\mathcal{S}$, all $t \in \mathcal{S}$, and all environments ρ we have: $t \in [\![\nu X.\varphi]\!]_\rho^\mathcal{T}$ iff there is a $T \subseteq \mathcal{S}$ s.t. $t \in T$ and $T \subseteq [\![\varphi]\!]_{\rho[X\mapsto T]}^\mathcal{T}$.*

PROOF Directly from the definition of the semantics. ■

Lemma 2 *For all φ, all transition systems $\mathcal{T}$ with state set $\mathcal{S}$, all $t \in \mathcal{S}$, and all environments ρ we have: $t \in [\![\neg\nu X.\varphi]\!]_\rho^\mathcal{T}$ iff for all $T \subseteq \mathcal{S}$: if $t \in T$ then there is a $u \in T$ s.t. $u \in [\![\neg\varphi]\!]_{\rho[X\mapsto T]}^\mathcal{T}$.*

PROOF This is a consequence of Lemma 1.

$$
\begin{array}{lll}
t \in [\![\neg\nu X.\varphi]\!]_\rho^\mathcal{T} & \text{iff} & t \notin [\![\nu X.\varphi]\!]_\rho^\mathcal{T} \\
 & \text{iff} & \nexists T \subseteq \mathcal{S} \text{ s.t. } t \in T \text{ and } T \subseteq [\![\varphi]\!]_{\rho[X\mapsto T]}^\mathcal{T} \\
 & \text{iff} & \forall T \subseteq \mathcal{S} : t \in T \Rightarrow T \not\subseteq [\![\varphi]\!]_{\rho[X\mapsto T]}^\mathcal{T} \\
 & \text{iff} & \forall T \subseteq \mathcal{S} : t \in T \Rightarrow \exists u \in T \text{ s.t. } u \notin [\![\varphi]\!]_{\rho[X\mapsto T]}^\mathcal{T} \\
 & \text{iff} & \forall T \subseteq \mathcal{S} : t \in T \Rightarrow \exists u \in T \text{ s.t. } u \in [\![\neg\varphi]\!]_{\rho[X\mapsto T]}^\mathcal{T}
\end{array}
$$

2 Model Checking as a Reachability Game

Let φ be closed and $\mathcal{T}$ be a transition system with state set $\mathcal{S}$. Configurations of the game $\mathcal{G}_\mathcal{T}(s,\varphi)$ for some $T \subseteq \mathcal{S}$ are of the form $\mathcal{S} \times (\mathcal{V} \cap FL(\varphi) \to 2^{\mathcal{S}}) \times FL(\varphi)$.

A configuration is written $t, \rho \vdash \psi$ and its intended meaning is $t \in [\![\psi]\!]^{\mathcal{T}}_{\rho}$. Let ρ_0 be some environment, for instance $\rho_0(X) = \emptyset$ for all $X \in \mathcal{V} \cap FL(\varphi)$. Every play of the game $\mathcal{G}_\mathcal{T}(s,\varphi)$ begins in the configuration $s, \rho_0 \vdash \varphi$. It is played between players $\exists$ and $\forall$ according to the following rules.

$$\frac{t,\rho \vdash \psi_1 \vee \psi_2}{t,\rho \vdash \psi_i} \ \exists i \in \{1,2\} \qquad \frac{t,\rho \vdash \neg(\psi_1 \vee \psi_2)}{t,\rho \vdash \neg\psi_i} \ \forall i \in \{1,2\} \qquad \frac{t,\rho \vdash \neg\neg\psi}{t,\rho \vdash \psi}$$

$$\frac{t,\rho \vdash \langle a \rangle \psi}{t',\rho \vdash \psi} \ \exists t \xrightarrow{a} t' \qquad \frac{t,\rho \vdash \neg\langle a \rangle \psi}{t,\rho \vdash \neg\psi} \ \forall t \xrightarrow{a} t'$$

$$\frac{t,\rho \vdash \nu X.\psi}{u,\rho[X \mapsto T] \vdash \psi} \ \exists T \supseteq \{t\}, \forall u \in T \qquad \frac{t,\rho \vdash \neg\nu X.\psi}{u,\rho[X \mapsto T] \vdash \neg\psi} \ \forall T \supseteq \{t\}, \exists u \in T$$

Player $\exists$ wins a play $C_0,\dots,C_n$ if

- $C_n = t,\rho \vdash q$, and $t \in L(q)$,
- $C_n = t,\rho \vdash X$, and $t \in \rho(X)$,
- $C_n = t,\rho \vdash [a]\psi$, and there is no $t' \in \mathcal{S}$ with $t \xrightarrow{a} t'$.

Player $\forall$ wins a play $C_0,\dots,C_n$ if

- $C_n = t,\rho \vdash q$, and $t \notin L(q)$,
- $C_n = t,\rho \vdash X$, and $t \notin \rho(X)$,
- $C_n = t,\rho \vdash \langle a \rangle\psi$, and there is no $t' \in \mathcal{S}$ with $t \xrightarrow{a} t'$.

Lemma 3 *Every play has a unique winner.*

PROOF There are no infinite plays since every rule decreases the formula component of the current configuration. Hence, every play must eventually reach a configuration with an atomic formula, or one of the players must get stuck beforehand. It is not hard to see that all cases are covered byt he 6 winning conditions, and that they are mutually exclusive. Note that the players cannot get stuck in the rules for fixpoint formulas. ■

Lemma 4 *For every transition system $\mathcal{T}$ with n states, every state s, and every closed formula φ: $\mathcal{G}_\mathcal{T}(s,\varphi)$ is a reachability game with at most $2n \cdot 2^{n \cdot var(\varphi)} \cdot |\varphi|$ many nodes.*

PROOF It should be clear from the definition of the winning conditions that $\mathcal{G}_{\mathcal{T}}(s, \varphi)$ is in fact a reachability game. Now consider the number of configurations of the form $t, \rho \vdash \psi$. Clearly, there are n different t and $|\varphi|$ many different ψ. The environment ρ is of type $\mathcal{V} \cap FL(\varphi) \to 2^{\mathcal{S}}$, i.e. it is a function of $var(\varphi)$ many arguments and 2^n many different values for each argument. Hence, there are only $2^{n \cdot var(\varphi)}$ many functions of that kind.

Finally, the additional factor 2 results from the transformation of $\mathcal{G}_{\mathcal{T}}(s, \varphi)$ into a directed graph in which every node represents the choice of one of the players. Note that consecutive choices in the rules for fixpoint formulas need to be transformed into several nodes in this graph. This, however, can at most double the number of nodes. ∎

Theorem 5 *Let φ be closed. Player $\exists$ wins the reachability game $\mathcal{G}_{\mathcal{T}}(s, \varphi)$ iff $s \in [\![\varphi]\!]^{\mathcal{T}}$.*

PROOF "⇒" By induction on the depth of the positional winning strategy as a tree. According to Lemma 3 this is finite and well-defined. We need to generalise the statement though. If player $\exists$ has a winning strategy for the game starting in position $t, \rho \vdash \psi$ then $t \in [\![\psi]\!]^{\mathcal{T}}_{\rho}$. For depth 0 the game starts in a configuration that is won by player $\exists$ according to one of his winning conditions. It is easy to see that here the claim holds.

For a depth greater than 0 the claim follows immediately from the inductive hypothesis and the semantics of $\mathcal{L}_{\mu}$. The only cases that are not straight-forward are those of the rules for the fixpoint formulas. However, they are proved in Lemmas 1 and 2.

"⇐" Again, we generalise the statement: if $t \in [\![\psi]\!]^{\mathcal{T}}_{\rho}$ then player $\exists$ has a winning strategy starting in the game position $t, \rho \vdash \psi$. Her strategy simply consists of preserving truth of a configuration along each play. It is not hard to see that she can do so and that player $\forall$ always preserves truth in whatever he chooses. The difficult cases are, again, the fixpoint formulas which are proved in Lemmas 1 and 2. According to Lemma 3, every play eventually reaches a position s.t. one of the winning conditions applies. If player $\exists$ has preserved this invariant then this is going to be a true configuration and it is easy to see that player $\forall$ cannot win a play ending in a true configuration. Hence, player $\exists$ wins it and her strategy is therefore a winning strategy. ∎

Theorem 6 *For all $k \in \mathbb{N}$, the model checking problem for each $\mathcal{L}^k_{\mu}$ on a fixed transition system can be solved in deterministic time $O(|\varphi|^2)$.*

PROOF Let $\mathcal{T}$ be a transition system with n states. Thm. 5 and Lemma 4 yield a reduction from the model checking problem for $\mathcal{L}^k_{\mu}$ to the problem of solving a reachability game of at most $2n \cdot 2^{nk} \cdot |\varphi|$ nodes and, thus, of at most $4n^2 \cdot 2^{2nk} \cdot |\varphi|^2$ many edges. It is well-known that reachability games can be solved in deterministic time linear in the number of their edges. If k and n are fixed then this is possible in time $O(|\varphi|^2)$. ∎

Remark The rules of the reachability games, in particular the ones for fixpoint formulas, are based on the translation of the modal μ-calculus into Monadic Second Order Logic (MSO) via

$$trans(\nu X.\varphi)_x \;:=\; \exists X.x \in X \wedge \forall y.y \in X \rightarrow trans(\varphi)_y$$

It is thus not surprising that the games can be generalised to reachability games for MSO showing that the expression complexity of MSO is also in P when the number of variables is fixed.